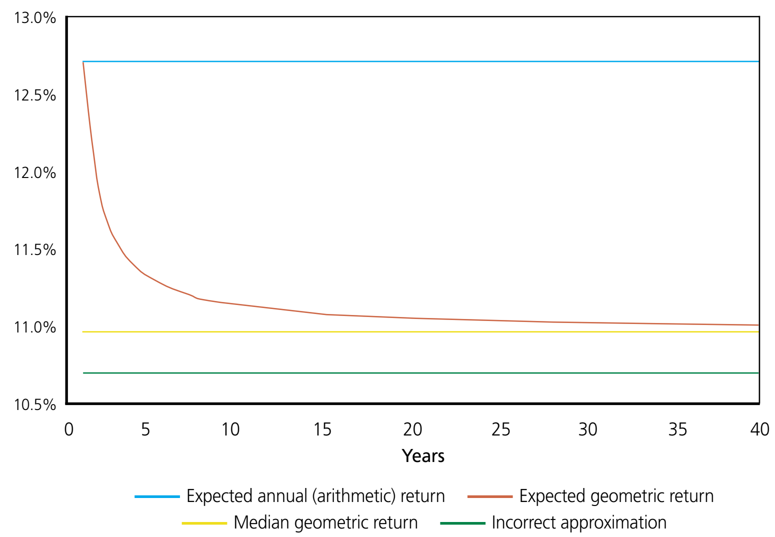
Expected geometric return and

Brian McCulloch tells us why we need to be careful when looking at prospective investment return measures.

Actuaries and other professional investment practitioners routinely report expected geometric return as a summary measure of the prospective performance of asset classes and investment portfolios. Expected geometric return has intuitive appeal because its historical counterpart, the geometric average, provides an annualised measure of the proportional change in wealth that actually occurred over a past time series, as if there had been no volatility in return. However, as a prospective measure, expected geometric return has limited value and often the expected annual (or arithmetic) return is actually a more relevant statistic for modelling and analysis. For example, the capital asset pricing model requires an unbiased estimate of the expected annual return. Similarly, projecting portfolio growth and discounting cashflows both also require the expected annual return.

Figure 1 Arithmetic and geometric measures of expected return



Despite this, the distinction between expected annual return and expected geometric return is not well understood, both in respect of individual asset classes and in respect of portfolios. This confusion persists even though it is explained routinely in finance textbooks and other reference sources. In order to help clear some of the confusion, this article examines some interesting facts about the properties of expected geometric return in portfolio analysis.

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Differing time horizons

If returns are assumed to follow a log-normal distribution and are serially independent, the exact relationship between the expected annual return ($E[r]$) and the path of the geometric mean ($E[g_N]$) over time (N) is:

$$E[g_N] = (1+E[r])\left(1+\frac{Var[r]}{(1+E[r])^2}\right)^{\frac{1-N}{2N}} - 1$$

For a given distribution of annual expected return and volatility, the geometric mean is smaller the longer the time period (N) that is being examined. This is illustrated in figure 1 using an expected annual return on investment of 12.7% and standard deviation of 20%. The expected annual return stays at 12.7%, regardless of what time horizon is being considered. However, the expected geometric return starts at 12.7% if the time horizon is one year, then declines, so that for a 20-year time horizon, the expected geometric return is about 11% per annum. A consequence of this is that if a long term (say 20 years) geometric return is reported, it will be relevant only to the specific time horizon to which it relates. In particular, it will understate expected geometric return over shorter periods and it will understate the expected annual return.

Using an approximation

There also is a frequently used approximation of the geometric mean. This is calculated as the expected annual (arithmetic) mean minus half its variance. This understates the true geometric mean for all time horizons, and it is especially wrong for shorter time horizons. This incorrect geometric approximation is also illustrated in the example in figure 1. If this approximation is used in a situation in which the annual expected value should have been used, the difference is even greater than if the correct calculation of the geometric mean had been used.

Another central measure of return that is often produced from modelling or simulations is the median. Under standard assumptions, the median geometric return is a constant that does not decline over time. It is also illustrated in figure 1. It is the asymptote that the expected geometric return converges to as the time horizon is expanded out infinitely.

Looking at portfolios

Asset classes combine into portfolios. The expected annual (arithmetic) portfolio return is a weighted average of the component assets' expected annual returns. However, it is a surprise to many practitioners who routinely focus on geometric return that the expected geometric portfolio return is not a weighted average of the component assets' expected geometric

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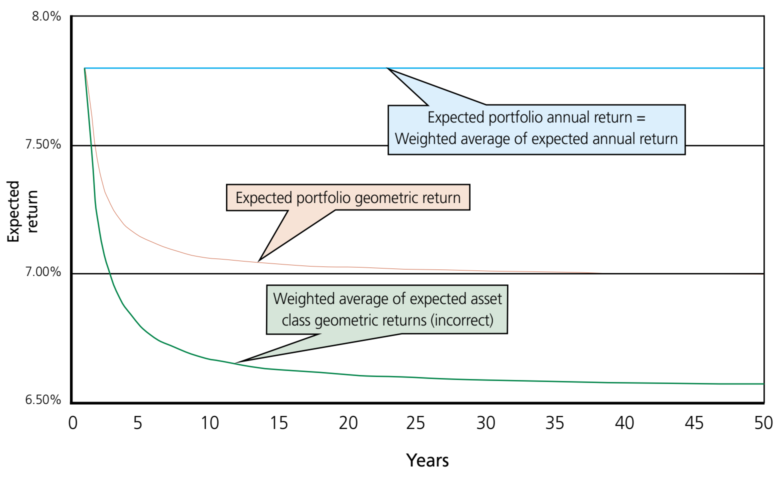
returns. A weighted average would understate the true expected geometric portfolio return. This can be shown using a result known as Jensen's Inequality. Instead, if we want to calculate the expected geometric return for a portfolio, it is necessary to take the weighted average of the component assets' expected annual (arithmetic) returns to get the expected portfolio annual arithmetic return, then use that result (along with the portfolio volatility and time horizon) to derive the expected portfolio geometric return for the required time horizon. Figure 2 illustrates how the correct and incorrect calculations of portfolio return can vary over time.

Since the incorrect calculation of the expected portfolio geometric return (using the weighted average of the component asset classes' geometric returns) understates the true portfolio expected geometric return, if this incorrect result is then used when the expected annual return would have been more appropriate, the difference is even more pronounced than if the correct geometric calculation had been used.

Figure 2 Alternative calculations of expected portfolio returns



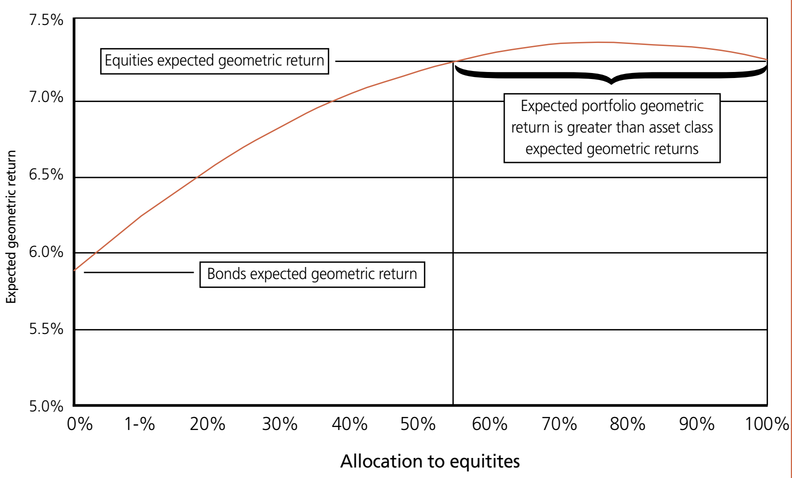
A surprising result

Another sometimes surprising outcome of the non-linear relationship between expected geometric asset class returns and expected portfolio geometric return is that it is possible for the expected portfolio geometric return to be greater than any of the individual asset class expected geometric returns.

Figure 3 shows how the portfolio geometric return of a two-asset portfolio, comprising bonds and equities, changes as the portfolio allocation moves from 0% in equities to 100% in equities. In this example, there is a region of portfolio composition, from 55% to 100% in equities, in which the portfolio geometric return is higher than that of either of the individual asset classes. Of course, this does not always happen – it depends on the structure of the return covariance matrix. Nonetheless, it is usual to find that the expected portfolio geometric return is at the upper end of the spread of the individual asset class expected geometric returns.

In conclusion, this article has examined a few of the properties of expected geometric return in portfolio analysis. As a purely descriptive measure of an observed time series of actual historical returns, the geometric average can be a useful summary measure of annualised return. However, for applications that involve future projections or other prospective analyses, the expected annual (arithmetic) return is usually the most relevant direct input to the calculations.

Figure 3 Expected portfolio geometric return



The expected geometric return understates this statistic and its frequently used approximation is worse. Even the supposedly straightforward calculation of weighted average portfolio return becomes somewhat complicated, and can produce counter-intuitive results if the focus of reporting is expected geometric return. ❑